

# ON THE $C^{1,\alpha}$ REGULARITY OF $p$ -HARMONIC FUNCTIONS IN THE HEISENBERG GROUP

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**ABSTRACT.** We present a proof of the local Hölder regularity of the horizontal derivatives of weak solutions to the  $p$ -Laplace equation in the Heisenberg group  $\mathbb{H}^1$  for  $p > 4$ .

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## 1. INTRODUCTION

We present a proof of the local Hölder regularity of derivatives of weak solutions to the  $p$ -Laplace equation in the Heisenberg group  $\mathbb{H}^1$  for the range  $p > 4$ . Our notation for the first Heisenberg group is  $\mathbb{H} = \mathbb{H}^1 = (\mathbb{R}^3, *)$ . Here, indicating points  $x, y \in \mathbb{H}$  by  $x = (x_1, x_2, z)$  and  $y = (y_1, y_2, s)$ , the group operation is

$$x * y = (x_1, x_2, z) * (y_1, y_2, s) = \left( x_1 + y_1, x_2 + y_2, z + s + \frac{1}{2}(x_1 y_2 - x_2 y_1) \right)$$

and a basis of left-invariant vector fields for the associated Lie algebra  $\mathfrak{h}$  is given by

$$X_1 = \partial_{x_1} - \frac{x_2}{2} \partial_z, \quad X_2 = \partial_{x_2} + \frac{x_1}{2} \partial_z \quad \text{and} \quad T = \partial_z.$$

If  $u : \Omega \rightarrow \mathbb{R}$  is a function from an open subset of  $\mathbb{H}$  we indicate by  $\nabla_{\mathbb{H}} u = (X_1 u, X_2 u)$  the horizontal gradient of  $u$ . We denote by  $HW^{1,p}(\Omega)$  the Sobolev space of functions  $u$  such that both  $u$  and  $\nabla_{\mathbb{H}} u \in L^p(\Omega)$ .

We study the regularity of solutions to the  $p$ -Laplace equation:

$$\sum_{i=1}^2 X_i \left( |\nabla_{\mathbb{H}} u|^{p-2} X_i u \right) = 0 \quad \text{in } \Omega. \quad (1.1)$$

The main result is the following:

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**Theorem 1.1.** Let  $u \in HW^{1,p}(\Omega)$  be a weak solution of the  $p$ -Laplace equation (1.1) for  $p > 4$  and let  $B_{R_0} \subset \subset \Omega$ . Then there exists  $\beta = \beta(p) \in (0, 1)$  such that for every  $l \in \{1, 2\}$  we have

$$\text{osc}_{B_R}(X_l u) \leq C_p \|\nabla_{\mathbb{H}} u\|_{L^\infty(B_{R_0})} \left(\frac{R}{R_0}\right)^\beta \quad \text{for all } R \leq R_0,$$

where  $C_p$  is a constant depending only on  $p$ .

In this work  $B_r(x_0)$  denotes a Carnot-Carathéodory ball of radius  $r$  and center  $x_0$  (we omit the center when it is not essential). We recall that the Carnot-Carathéodory distance between  $x$  and  $y \in \mathbb{H}$  is defined as

$$d_{cc}(x, y) = \inf\{l(\Gamma) \mid \Gamma \in S(x, y)\}.$$

Here  $S(x, y)$  denotes the set of all horizontal subunitary curves joining  $x$  and  $y$ , i.e. absolutely continuous curves  $\Gamma : [0, T] \rightarrow \Omega$  such that  $\Gamma'(t) = \sum_{j=1}^2 \alpha_j(t) X_j(\Gamma(t))$  for some real valued functions  $\alpha_j$  with  $\sum_{j=1}^2 \alpha_j(t)^2 \leq 1$ . The length of such a curve is defined to be  $l(\Gamma) = T$ .

To prove regularity results in general one considers a family of approximated non degenerate problems and tries to produce estimates independent of the non degeneracy parameter, in such a way that they can be applied to the degenerate equation by passing to the limit. More precisely, here we consider the non degenerate equations

$$\sum_{i=1}^2 X_i \left( (\delta^2 + |\nabla_{\mathbb{H}} u|^2)^{\frac{p-2}{2}} X_i u \right) = 0 \quad \text{in } \Omega \quad (1.2)$$

for a parameter  $\delta > 0$ . Equation (1.1) corresponds to the degenerate case  $\delta = 0$ .

The Heisenberg group presents new challenges with respect to its Euclidean counterpart, since we only assume that  $u$  is in the horizontal Sobolev space  $HW^{1,p}$  and differentiating the equation produces terms involving the vertical derivative  $Tu$ , due to the non commutativity of the horizontal vector fields. This constitutes the main difficulty.

For  $p = 2$  it is now classical that the solutions of equation (1.1) are  $C^\infty$  [14]. For  $p \neq 2$  the Hölder regularity of solutions of equations modeled on (1.2) was established by Capogna and Garofalo [3] and Lu [18]. Later Manfredi and Mingione [20] were able to prove  $C^{1,\alpha}$  regularity in the non degenerate case for  $2 \leq p < c(n) < 4$  and by adapting an argument used by Capogna they achieve  $C^\infty$  regularity for this range of values of  $p$ . The starting point is the integrability result for the vertical derivative  $Tu \in L^p$  established by Domokos for  $1 < p < 4$  in [7], where he extends integrability results considered by Marchi for  $1 + \frac{1}{\sqrt{5}} < p < 1 + \sqrt{5}$  in [21], [22].

Mingione, Zatorska-Goldstein and Zhong proved in [23] that the Euclidean gradient of solutions to the non degenerate equation are  $C^{1,\alpha}$  for  $2 \leq p < 4$  and also that solutions to the degenerate equation are locally Lipschitz continuous for  $2 \leq p < 4$ .

Zhong in [26] extended the Hilbert-Haar theory to the Heisenberg group setting and proved that solutions to the degenerate equation (1.1) are locally Lipschitz for the full range  $1 < p < \infty$ . For an account of this theory, further historical details and additional references see [24].

As for the Hölder continuity of the horizontal derivatives for the degenerate equation (1.1) the only published result for  $p \neq 2$  has been obtained by Manfredi and Domokos in

[9], [8] via the Cordes perturbation technique for  $p$  near 2.

The proof of the Hölder continuity of the horizontal derivatives contained in this work uses the particular form of the equation in  $\mathbb{H}^1$  and new integration by parts for the second derivatives that produce weights of the form  $(\delta^2 + |\nabla_{\mathbb{H}} u|)^{\frac{p-4}{2}}$ . This is the reason why our proof is only valid in the first Heisenberg group  $\mathbb{H}^1$  and for the range  $p > 4$ .

## 2. PRELIMINARIES

**2.1. The  $p$ -Laplace Equation.** We will consider the non degenerate  $p$ -Laplace equation (1.2). Denoting  $z = (z_1, z_2) \in \mathbb{R}^2$  and calling

$$a_i(z) = (\delta^2 + |z|^2)^{\frac{p-2}{2}} z_i \quad \text{and} \quad w = \delta^2 + |\nabla_{\mathbb{H}} u|^2,$$

equation (1.2) rewrites as

$$\sum_{i=1}^2 X_i a_i(\nabla_{\mathbb{H}} u) = 0 \quad \text{in } \Omega \quad (2.1)$$

and satisfies the following ellipticity and growth conditions for all  $p > 1$ :

$$\begin{aligned} \sum_{i,j=1}^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) \xi_i \xi_j &\geq c_p w^{\frac{p-2}{2}} |\xi|^2, \\ |a_i(\nabla_{\mathbb{H}} u)| &\leq w^{\frac{p-1}{2}}, \\ |\partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| &\leq C_p w^{\frac{p-2}{2}}, \\ |\partial_{z_s} \partial_{z_j} a_i(\nabla_{\mathbb{H}} u)| &\leq C_p w^{\frac{p-3}{2}} \end{aligned} \quad (2.2)$$

and

$$\frac{\partial_{z_j} a_i(z)}{\partial_{z_l} a_l(z)} \leq C_p \quad \text{for all } i, j, l \in \{1, 2\}. \quad (2.3)$$

We remark that the proofs presented in this work depend only on these properties, therefore they extend to more general equations of  $p$ -Laplacean type as in (2.1) for  $a_i$  of class  $C^2$  satisfying (2.2) and (2.3).

We say that a function  $u \in HW^{1,p}(\Omega)$  is a weak solution of (1.2) if

$$\int_{\Omega} w^{\frac{p-2}{2}} \langle \nabla_{\mathbb{H}} u, \nabla_{\mathbb{H}} \varphi \rangle \, dx = 0 \quad \text{for all } \varphi \in HW_0^{1,p}(\Omega), \quad (2.4)$$

where  $HW_0^{1,p}(\Omega)$  is the closure of the space of  $C^\infty$  compactly supported functions with respect to the Horizontal Sobolev norm.

**2.2. Previous Results.** We now collect some known results about the non degenerate equation (1.2) that will be used in the following sections. We refer to [24] for a detailed presentation and complete proofs.

First we have that solutions to the non degenerate  $p$ -Laplace equation (1.2) are smooth. This was proved by Capogna in [2] for  $p \geq 2$  and extended to the full range  $1 < p < \infty$  in [24] Chapter 4 by adapting techniques of Domokos in [7].

As a consequence we have

$$\sum_{i,j=1}^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_i X_j u = 0 \quad \text{a.e. in } \Omega \quad (2.5)$$

hence we can express  $X_1 X_1 u$  (respectively  $X_2 X_2 u$ ) in terms of  $X_i X_j u$  where at least one index is a 2 (respectively a 1). This will be a crucial point later.

We now collect the equations satisfied by the horizontal and vertical derivatives ( see [24], Lemma 4.1):

**Lemma 2.1.** The functions  $X_1u$ ,  $X_2u$  and  $Tu$  are weak solutions respectively of the following equations (in  $\Omega$ ):

$$\sum_{i=1}^2 X_i \left( \sum_{j=1}^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}u}) X_j X_1 u \right) + \sum_{i=1}^2 X_i (\partial_{z_2} a_i(\nabla_{\mathbb{H}u}) Tu) + T(a_2(\nabla_{\mathbb{H}u})) = 0 \quad (2.6)$$

$$\sum_{i=1}^2 X_i \left( \sum_{j=1}^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}u}) X_j X_2 u \right) - \sum_{i=1}^2 X_i (\partial_{z_1} a_i(\nabla_{\mathbb{H}u}) Tu) - T(a_1(\nabla_{\mathbb{H}u})) = 0 \quad (2.7)$$

$$\sum_{i=1}^2 X_i \left( \sum_{j=1}^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}u}) X_j Tu \right) = 0. \quad (2.8)$$

In [26] Zhong established the weighted higher order integrability of  $Tu$  as follows:

**Lemma 2.2.** For all  $q > 4$  and  $\xi \in C_0^\infty(\Omega)$  we have:

$$\int_{\Omega} \xi^q w^{\frac{p-2}{2}} |Tu|^q dx \leq C(q) \left( \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \|\xi T\xi\|_{L^\infty} \right)^{\frac{q}{2}} \int_{\text{supp}(\xi)} w^{\frac{p-2+q}{2}} dx, \quad (2.9)$$

where  $C(q) = C_p^{\frac{q-2}{2}} q^{q+8}$  and  $C_p$  depends only on  $p$ .

For the sake of completeness we give a proof in the Appendix.

### 3. DE GIORGI CLASSES IN THE HEISENBERG GROUP

We now describe a type of De Giorgi class in the Heisenberg group. These kind of spaces were introduced and studied by De Giorgi in the Euclidean case (see [5]). We will use the standard notation for super- (sub-) level sets of a measurable function

$$A_{k,r}^+ = A_{k,r}^+(f) = B_r \cap \{f > k\},$$

$$A_{k,r}^- = A_{k,r}^-(f) = B_r \cap \{f < k\}.$$

**Definition 3.1** (De Giorgi class in the Heisenberg group). Let  $\Omega \subset \mathbb{H}$  be open,  $\gamma, \chi$  positive real constants and  $q > Q$ . A function  $f \in HW_{\text{loc}}^{1,2}(\Omega) \cap L_{\text{loc}}^\infty(\Omega)$  belongs to the De Giorgi class  $DG^+(\Omega, \gamma, \chi, q)$  if

$$\int_{A_{h,r'}^+} |\nabla_{\mathbb{H}} f|^2 dx \leq \frac{\gamma}{(r-r')^2} \sup_{B_r} |(f-h)^+|^2 |A_{k,r}^+| + \chi |A_{k,r}^+|^{1-\frac{2}{q}} \quad (3.1)$$

for some concentric balls  $B_{r'} \subset B_r \subset \subset \Omega$  and levels  $h \in \mathbb{R}$ .

In this section we consider an arbitrary ball  $B_R \subset \subset \Omega$  and denote by  $M = M(R) = \sup_{B_R} f$  and  $m = m(R) = \inf_{B_R} f$ .

We are taking the following Lemma from [15], Lemma 2.3, where it is proved in a more general setting.

**Lemma 3.2.** Let  $l > k$ ,  $f \in HW_{\text{loc}}^{1,1}(\Omega)$ ,  $B_r \subset \subset \Omega$ . Then if  $|B_r \setminus A_{k,r}^+| > 0$  we have:

$$(l-k) |A_{l,r}^+|^{1-\frac{1}{Q}} \leq \frac{C(Q)r^Q}{|B_r \setminus A_{k,r}^+|} \int_{A_{k,r}^+ \setminus A_{l,r}^+} |\nabla_{\mathbb{H}} f| dx, \quad (3.2)$$

where  $C(Q)$  is a constant depending only on  $Q$ .

The next Lemma is adapted from Lemma 2.3 in [19] and Lemma 6.1 in [16].

**Lemma 3.3.** Let  $0 < \lambda_0, \lambda_1 < 1$  and  $k < M$ . Suppose  $f \in DG^+(\Omega, \gamma, \chi, q)$  for  $h \in [k, \lambda_0 k + (1 - \lambda_0)M]$  and for  $r' < r \in [\lambda_1 R, R]$ . Then there exists  $\theta = \theta(Q, \gamma, \lambda_0, \lambda_1) \in (0, 1)$  such that if

$$M - k \geq \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}},$$

then

$$|A_{k,R}^+| \leq \theta |B_R| \quad \text{implies} \quad f \leq \lambda_0 k + (1 - \lambda_0)M \quad \text{a.e. in } B_{\lambda_1 R}.$$

The following Lemma is adapted from Lemma 2.4 in [19] and Lemma 6.2 in [16].

**Lemma 3.4.** Let  $0 < \lambda_1 < 1$  and  $k < M$ . Suppose  $f \in DG^+(\Omega, \gamma, \chi, q)$  for  $h \in [k, M]$  and for  $r' = \lambda_1 R, r = R$ . If there exists a constant  $0 < C_0 < 1$  such that  $|A_{k,\lambda_1 R}^+| \leq C_0 |B_{\lambda_1 R}|$  then given  $0 < \theta < 1$  there exists  $s = s(Q, \gamma, \lambda_1, C_0, \theta) \in \mathbb{N}$  such that

$$\text{if } M - k \geq 2^s \chi^{\frac{1}{2}} R^{1 - \frac{Q}{q}} \quad \text{then} \quad |A_{k_s, \lambda_1 R}^+| \leq \theta |B_{\lambda_1 R}|,$$

where  $k_s = k + (1 - 2^{-s})(M - k)$  is a level set between  $k$  and  $M$ .

Combining the previous Lemmas we get an estimate for the decay of the oscillation of functions in the De Giorgi class. We are adapting it from Lemma 2.5 in [19] and from [16].

**Lemma 3.5** (Oscillation estimate). Let  $0 < \lambda_1 < 1$  and suppose that for radii  $r' < r \in [\lambda_1 R, R]$  we have  $f \in DG^+(\Omega, \gamma, \chi, q)$  for  $h \in [\frac{m+M}{2}, M]$  and  $-f \in DG^+(\Omega, \gamma, \chi, q)$  for  $h \in [-M, -\frac{m+M}{2}]$ . Then there exists  $A = A(Q, \gamma, \lambda_1) \in (0, 1)$  such that

$$\text{osc}_{B_{\lambda_1 R}} f \leq A \text{osc}_{B_R} f + B R^{1 - \frac{Q}{q}}, \quad (3.3)$$

where

$$B = \frac{\chi^{\frac{1}{2}}}{4(1 - A)}.$$

#### 4. MAIN ESTIMATE

From now on we will fix a ball  $B_{R_0} \subset\subset \Omega$  and for a concentric ball  $B_R \subset B_{R_0}$  we introduce the notation

$$\mu(R) = \max_{1 \leq l \leq 2} \|X_l u\|_{L^\infty(B_R)} \quad \text{and} \quad \lambda(R) = \frac{1}{2} \mu(R).$$

In this section we prove the following Proposition which contains the main estimates and constitutes the novel contribution of this work:

**Proposition 4.1.** Let  $B_{R_0} \subset\subset \Omega$  and let  $u \in HW^{1,p}(\Omega)$  be a weak solution of the non degenerate equation (1.2) for  $p > 4$ . For every  $0 < r' < r < \frac{R_0}{2}$ ,  $l = 1, 2$  and for every  $q > \max\{4, 2 + \frac{4}{p-4}\}$  we have :

$$\int_{B_{r'}} w^{\frac{p-2}{2}} |\nabla_H(X_l u - k)^+|^2 dx \leq \frac{C_p}{(r - r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^+|^2 dx + \chi |A_{k,r}^+(X_l u)|^{1 - \frac{2}{q}}, \quad (4.1)$$

$$\int_{B_{r'}} w^{\frac{p-2}{2}} |\nabla_H(X_l u - k)^-|^2 dx \leq \frac{C_p}{(r - r')^2} \int_{B_r} w^{\frac{p-2}{2}} |(X_l u - k)^-|^2 dx + \chi |A_{k,r}^-(X_l u)|^{1 - \frac{2}{q}}. \quad (4.2)$$

The inequalities (4.1) hold for levels  $k \geq -\mu(R_0)$ , while (4.2) hold for levels  $k \leq \mu(R_0)$ . The constant  $C_p$  depends only on  $p$  and the parameter  $\chi$  is given by

$$\chi = \frac{C_p q^6}{R_0^2} (\delta^2 + \mu(R_0)^2)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}}. \quad (4.3)$$

*Proof.* We will prove (4.1) for  $l = 1$ , the other estimates follow in a similar fashion. We use the notation  $v_l = (X_l u - k)^+$ . Fix  $0 < r' < r < \frac{R_0}{2}$  and let  $\phi = \xi^2 v_1$ , where  $\xi$  is a cut-off function between  $B_{r'}$  and  $B_r$  with  $|\nabla_{\mathbb{H}} \xi| \leq \frac{C}{(r-r')}$ . Denote  $A_{k,r}^+(X_1 u)$  for simplicity by  $A_{k,r}^+$  and we use the usual convention of sum on repeated indices. Test equation (2.6) with  $\phi$  to get:

$$\begin{aligned} J_1 &= \int_{B_r} \xi^2 \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_1 u X_i v_1 \, dx = -2 \int_{B_r} \xi \partial_{z_j} a_i(\nabla_{\mathbb{H}} u) X_j X_1 u X_i \xi v_1 \, dx \\ &\quad - \int_{B_r} \xi^2 \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) X_i v_1 T u \, dx \\ &\quad - 2 \int_{B_r} \xi \partial_{z_2} a_i(\nabla_{\mathbb{H}} u) X_i \xi T u v_1 \, dx \\ &\quad - \int_{B_r} a_2(\nabla_{\mathbb{H}} u) T(\xi^2 v_1) \, dx \\ &= J_2 + J_3 + J_4 + J_5. \end{aligned}$$

Routine calculations using Young's inequality and (2.2) allow to estimate  $J_i$  for  $1 \leq i \leq 4$  as follows

$$\int_{B_r} \xi^2 w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_1|^2 \, dx \leq C \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2 w^{\frac{p-2}{2}} v_1^2 \, dx + C \int_{A_{k,r}^+} \xi^2 w^{\frac{p-2}{2}} |T u|^2 \, dx + J_5. \quad (4.4)$$

The new idea is to estimate  $J_5$  by integrating by parts twice

$$\begin{aligned} J_5 &= \int_{B_r} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u) X_j T u \xi^2 v_1 \, dx = - \int_{B_r} T u X_j (\partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \xi^2 v_1) \, dx \\ &= - \int_{B_r} T u \partial_{z_s} \partial_{z_j} a_2(\nabla_{\mathbb{H}} u) X_j X_s u \xi^2 v_1 \, dx \\ &\quad - 2 \int_{B_r} T u \partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \xi X_j \xi v_1 \, dx \\ &\quad - \int_{B_r} T u \partial_{z_j} a_2(\nabla_{\mathbb{H}} u) \xi^2 X_j v_1 \, dx \\ &= J_{5,1} + J_{5,2} + J_{5,3}. \end{aligned}$$

Note that  $J_{5,2}$  and  $J_{5,3}$  can be estimated respectively as  $J_4$  and  $J_3$ .

Denoting  $J_{5,1} = \sum_{s,j} J_{5,1}^{s,j}$  we have:

$$\begin{aligned} \sum_j J_{5,1}^{1,j} &\leq C \int_{B_r} \xi^2 w^{\frac{p-3}{2}} |\nabla_{\mathbb{H}} v_1| v_1 |T u| \, dx \leq C \varepsilon \int_{B_r} \xi^2 w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_1|^2 \, dx + \frac{C}{\varepsilon} \int_{B_r} \xi^2 w^{\frac{p-4}{2}} |T u|^2 v_1^2 \, dx \\ J_{5,1}^{2,1} &= \int_{B_r} \partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u) X_1 X_2 u T u \xi^2 v_1 \, dx = \int_{B_r} \partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u) X_2 X_1 u T u \xi^2 v_1 \, dx \\ &\quad + \int_{B_r} \partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u) |T u|^2 \xi^2 v_1 \, dx. \end{aligned}$$

The first term of the last equality has the same estimate as  $J_{5,1}^{1,j}$ . For the other term we have:

$$\begin{aligned} \int_{B_r} \partial_{z_2} \partial_{z_1} a_2(\nabla_{\mathbb{H}} u) |T u|^2 \xi^2 v_1 \, dx &\leq C \int_{B_r} \xi^2 w^{\frac{p-3}{2}} |T u|^2 v_1 \, dx \leq C \int_{A_{k,r}^+} \xi^2 w^{\frac{p-2}{2}} |T u|^2 \, dx \\ &\quad + C \int_{B_r} \xi^2 w^{\frac{p-4}{2}} |T u|^2 v_1^2 \, dx. \end{aligned}$$

Now another key step is to use the equation in (2.5) and (2.3) to get:

$$\begin{aligned} J_{5,1}^{2,2} &= \int_{B_r} \partial_{z_2} \partial_{z_2} a_2(\nabla_{\mathbb{H}} u) X_2 X_2 u \xi^2 v_1 T u \, dx \leq C_p \int_{B_r} \partial_{z_2} \partial_{z_2} a_2(\nabla_{\mathbb{H}} u) X_1 X_1 u T u \xi^2 v_1 \, dx \\ &\quad + C_p \int_{B_r} \partial_{z_2} \partial_{z_2} a_2(\nabla_{\mathbb{H}} u) X_2 X_1 u T u \xi^2 v_1 \, dx \\ &\quad + C_p \int_{B_r} \partial_{z_2} \partial_{z_2} a_2(\nabla_{\mathbb{H}} u) X_1 X_2 u T u \xi^2 v_1 \, dx \\ &= F_1 + F_2 + F_3. \end{aligned}$$

Note that  $F_1$  and  $F_2$  can be estimated as  $J_{5,1}^{1,j}$  while  $F_3$  can be estimated as  $J_{5,1}^{2,1}$ .

Choosing  $\varepsilon$  small enough (4.4) becomes:

$$\begin{aligned} \int_{B_r} \xi^2 w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_1|^2 \, dx &\leq C \int_{B_r} |\nabla_{\mathbb{H}} \xi|^2 w^{\frac{p-2}{2}} v_1^2 \, dx + C \int_{A_{k,r}^+} \xi^2 w^{\frac{p-2}{2}} |T u|^2 \, dx + C \int_{B_r} \xi^2 w^{\frac{p-4}{2}} |T u|^2 v_1^2 \, dx \\ &= I_1 + I_2 + I_3. \end{aligned} \tag{4.5}$$

We only need to estimate  $I_2$  and  $I_3$ . We use Hölder's inequality with exponent  $q/2$  and Lemma 2.2:

$$\begin{aligned} I_2 &\leq \left( \int_{A_{k,r}^+} \xi^q w^{\frac{p-2}{2}} |T u|^q \, dx \right)^{\frac{2}{q}} \left( \int_{A_{k,r}^+} w^{\frac{p-2}{2}} \, dx \right)^{1-\frac{2}{q}} \\ &\leq \left( \int_{B_{R_0}} \eta^q w^{\frac{p-2}{2}} |T u|^q \, dx \right)^{\frac{2}{q}} \left( \int_{A_{k,r}^+} w^{\frac{p-2}{2}} \, dx \right)^{1-\frac{2}{q}} \\ &\leq \left( \left( \|\nabla_{\mathbb{H}} \eta\|_{L^\infty}^2 + \|\eta T \eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \, dx \right)^{\frac{2}{q}} (\delta^2 + \mu(r)^2)^{\frac{p-2}{2}(1-\frac{2}{q})} |A_{k,r}^+|^{1-\frac{2}{q}} \\ &\leq \frac{C_p q^6}{R_0^2} (\delta^2 + \mu(R_0)^2)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} |A_{k,r}^+|^{1-\frac{2}{q}}, \end{aligned}$$

where  $\eta$  is a cut-off function between  $B_{\frac{R_0}{2}}$  and  $B_{R_0}$  with  $|\nabla_{\mathbb{H}} \eta| \leq \frac{C}{R_0}$ . In a similar way and noting that  $v_1^2 \leq 2(\delta^2 + \mu(R_0)^2)$  for  $k \geq -\mu(R_0)$  we get:

$$\begin{aligned} I_3 &\leq (\delta^2 + \mu(R_0)^2) \left( \int_{A_{k,r}^+} \xi^q w^{\frac{p-2}{2}} |T u|^q \, dx \right)^{\frac{2}{q}} \left( \int_{A_{k,r}^+} w^{\frac{p-4}{2} - \frac{2}{q-2}} \, dx \right)^{1-\frac{2}{q}} \\ &\leq (\delta^2 + \mu(R_0)^2) \left( \left( \|\nabla_{\mathbb{H}} \eta\|_{L^\infty}^2 + \|\eta T \eta\|_{L^\infty} \right)^{\frac{q}{2}} \int_{B_{R_0}} w^{\frac{p-2+q}{2}} \, dx \right)^{\frac{2}{q}} (\delta^2 + \mu(r)^2)^{\left(\frac{p-4}{2} - \frac{2}{q-2}\right)(1-\frac{2}{q})} |A_{k,r}^+|^{1-\frac{2}{q}} \\ &\leq \frac{C_p q^6}{R_0^2} (\delta^2 + \mu(R_0)^2)^{\frac{p}{2}} |B_{R_0}|^{\frac{2}{q}} |A_{k,r}^+|^{1-\frac{2}{q}}. \end{aligned}$$

■

**Remark 4.2.** Note that the main difficulty in the proof is estimating the terms containing  $Tu$ . In particular in  $J_5$  we integrate by parts twice to avoid dealing with terms involving  $\nabla_{\mathbb{H}} T u$ . Then we use the equation in order to estimate terms with  $X_2 X_2 u$  appropriately with quantities independent of  $\delta$  or that can be absorbed in the right hand side.

## 5. OSCILLATION ESTIMATE

In this Section we prove our main result Theorem 1.1. Recall that we fixed a ball  $B_{R_0} \subset\subset \Omega$  and we now consider an arbitrary concentric ball  $B_R \subset B_{\frac{R_0}{2}}$ .

**Remark 5.1.** Let  $u \in HW^{1,p}(\Omega)$  be a weak solution of the non degenerate equation (1.2) for  $p > 4$ . For  $\delta \geq \lambda(R)$  we easily get that for every  $\lambda_1 \in (0, 1)$  there exists  $A = A(p, \lambda_1)$  such that

$$\text{osc}_{B_{\lambda_1 R}}(X_l u) \leq A \text{osc}_{B_R}(X_l u) + BR^\alpha \quad \text{for every } l \in \{1, 2\},$$

where

$$B = \frac{C_p q^{\frac{6}{p}} (\delta^2 + \mu(R_0)^2)^{\frac{1}{2}}}{4(1-A)R_0^\alpha} \quad \text{and} \quad \alpha = \left(1 - \frac{Q}{q}\right) \frac{2}{p}.$$

*Proof.* Since  $\delta \geq \lambda(R)$  we can get rid of the weight and obtain that  $X_l u$  is in a De Giorgi class. Indeed from (4.1) we get

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, dx \leq \frac{C_p}{(r-r')^2} \int_{B_r} v_l^2 \, dx + \frac{2^{p-2} \chi}{\mu(R)^{p-2}} |A_{k,r}^+(X_l u)|^{1-\frac{2}{q}}$$

for all levels  $k > -\mu(R_0)$  and radii  $r' < r < R$ . Now if

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{(1-\frac{Q}{q})\frac{2}{p}} \tag{5.1}$$

denoting by

$$\chi' = C_p q^{\frac{12}{p}} (\delta^2 + \mu(R_0)^2) \left(\frac{R}{R_0}\right)^{2(1-\frac{Q}{q})\frac{2}{p}} R^{2(\frac{Q}{q}-1)}$$

we get that  $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$  for all levels  $k > -\mu(R_0)$  and radii  $r' < r < R$ .

Analogously from (4.2) we get also  $-X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$  for all levels  $k < \mu(R_0)$  and radii  $r' < r < R$ , hence we can apply the Oscillation estimate in Theorem 3.5 to get for any  $\lambda_1 \in (0, 1)$  the existence of  $A = A(p, \lambda_1) \in (0, 1)$  such that for every  $l \in \{1, 2\}$  we have

$$\text{osc}_{B_{\lambda_1 R}}(X_l u) \leq A \text{osc}_{B_R}(X_l u) + B' R^{1-\frac{Q}{q}},$$

where  $4(1-A)B' = (\chi')^{\frac{1}{2}}$ . By the definition of  $\chi'$ , and combining with the case when (5.1) does not hold, we get the result.  $\blacksquare$

We now consider the interesting case when the equation degenerates, namely  $\delta < \lambda(R)$ . Here we face an alternative: either the maximum  $\mu(R)$  has the right 'Hölder decay' or the horizontal gradient  $\nabla_{\mathbb{H}} u$  is bounded away from zero, and hence the equation behaves like the non degenerate case in Remark 5.1. More precisely we have:

**Proposition 5.2.** Let  $u \in HW^{1,p}(\Omega)$  be a weak solution of the non degenerate equation (1.2) for  $p > 4$  and consider  $B_R \subset B_{\frac{R_0}{2}}$ . Then there exist  $\theta = \theta(p) \in (0, 1)$  and  $A = A(p) \in (0, 1)$  such that:

**Case 1.** If for some  $l \in \{1, 2\}$  we have either

$$\left| B_R \cap \left\{ X_l \geq \frac{1}{2} \mu(R) \right\} \right| \geq \theta |B_R| \tag{5.2}$$

or

$$\left| B_R \cap \left\{ X_l \leq -\frac{1}{2} \mu(R) \right\} \right| \geq \theta |B_R| \tag{5.3}$$

then

$$\text{either } \mu(R) \leq c_p \chi^{\frac{1}{p}} R^{(1-\frac{Q}{q})\frac{2}{p}} \quad \text{or} \quad |X_l u| \geq \frac{1}{32} \mu(R) \quad \text{a.e in } B_{R/2}$$



where  $c_p = 2(4/3)^{\frac{2}{p}}$ .

**Case 2.** If for every  $l \in \{1, 2\}$  neither (5.2) nor (5.3) holds then

$$\mu(R/2) \leq A\mu(R) + BR^\alpha, \quad (5.4)$$

where

$$B = \frac{C_p q^{\frac{6}{p}}}{2(1-A)} \frac{\mu(R_0)}{R_0^\alpha} \quad \text{and} \quad \alpha = \left(1 - \frac{Q}{q}\right) \frac{2}{p}.$$

*Proof. Case 1.*

Consider (5.3). We will show that it implies  $X_l u \leq -\frac{1}{32}\mu(R)$  provided  $\mu(R) \geq c_p \chi^{\frac{1}{p}} R^{(1-\frac{Q}{q})\frac{2}{p}}$ . Define the auxiliary function

$$V_l = |X_l u|^{\frac{p}{2}} \text{sign}(X_l u).$$

Observe that  $|V_l| \leq (2\lambda(R))^{\frac{p}{2}}$  on  $B_R$ . Also

$$|\nabla_{\mathbb{H}} V_l|^2 = \frac{p^2}{4} |X_l u|^{p-2} |\nabla_{\mathbb{H}} X_l u|^2. \quad (5.5)$$

Denote by  $h = |k|^{\frac{p}{2}} \text{sign}(k) = g(k)$  and note that  $\{X_l > k\} = \{V_l > h\}$  therefore (4.1) becomes:

$$\begin{aligned} \int_{A_{h,r'}^+(V_l)} |\nabla_{\mathbb{H}} V_l|^2 dx &\leq C_p \int_{B_{r'}} w^{\frac{p-2}{2}} |\nabla_{\mathbb{H}} v_l|^2 dx \leq \frac{C_p}{(r-r')^2} (\mu(r) - k)^2 (\delta^2 + \mu(r)^2)^{\frac{p-2}{2}} |A_{h,r}^+(V_l)| \\ &\quad + \chi |A_{h,r}^+(V_l)|^{1-\frac{2}{q}} \\ &\leq \frac{C_p}{(r-r')^2} (\lambda(R))^p |A_{h,r}^+(V_l)| + \chi |A_{h,r}^+(V_l)|^{1-\frac{2}{q}} \end{aligned} \quad (5.6)$$

for  $k > -\lambda(R)$  and  $r' < r \leq R$ . Denoting by  $H = H(R) = (\lambda(R))^{\frac{p}{2}}$  the inequality (5.6) rewrites as:

$$\int_{A_{h,r'}^+(V_l)} |\nabla_{\mathbb{H}} V_l|^2 dx \leq \frac{C_p}{(r-r')^2} (H(R))^2 |A_{h,r}^+(V_l)| + \chi |A_{h,r}^+(V_l)|^{1-\frac{2}{q}} \quad (5.7)$$

for levels  $h > -H(R)$  and radii  $r' < r \leq R$ . Now denote  $M(\frac{R}{2}) = \sup_{B_{R/2}} V_l$ .

**Case a.**  $M(\frac{R}{2}) < -\frac{H(R)}{4}$ .

This means  $X_l u < 0$  in  $B_{R/2}$  and after some algebraic manipulations

$$X_l u < -\frac{\mu(R)}{32} \quad \text{on } B_{R/2}.$$

**Case b.**  $M(\frac{R}{2}) \geq -\frac{H(R)}{4}$ .

For levels  $h \in [-H(R), -H(R)/2]$  we have  $\sup_{B_{R/2}} (V_l - h)^+ \geq \frac{H(R)}{4}$ , therefore

$$\frac{(H(R))^2}{16} \leq \sup_{B_r} |(V_l - h)^+|^2 \quad \text{for } r \in [R/2, R].$$

Hence, from (5.7) we get that  $V_l \in DG^+(B_{R_0}, C_p, \chi, q)$  for levels  $h \in [-H(R), -H(R)/2]$  and radii  $r' < r \in [R/2, R]$ . Apply Lemma 3.3 with  $k = -H(R)$ ,  $\lambda_0 = \frac{2^{\frac{p}{2}+1/2}}{2^{\frac{p}{2}+1}}$ ,  $\lambda_1 = 1/2$  to get the existence of  $\theta_1 = \theta_1(p) \in (0, 1)$  such that  $|A_{-H(R)}^+(V_l)| \leq \theta_1 |B_R|$  implies  $V_l \leq -\frac{H(R)}{2}$  on  $B_{R/2}$ , provided  $M(R) + H(R) \geq \chi^{\frac{1}{2}} R^{1-\frac{Q}{q}}$ . This is true if

$$\mu(R) \geq 2 \left(\frac{4}{3}\right)^{\frac{2}{p}} \chi^{\frac{1}{p}} R^{(1-\frac{Q}{q})\frac{2}{p}}. \quad (5.8)$$

Then as in Case a, we obtain

$$X_l u < -\frac{\mu(R)}{8} \quad \text{in } B_{R/2}.$$

Observe that  $\{V_l > -H(R)\} = \{X_l u > -\lambda(R)\} = \{X_l u > -\mu(R)/2\}$ . Passing to the complements we have proved that there exists  $\theta = 1 - \theta_1$  such that (5.3) implies  $X_l u \leq -\mu(R)/32$  on  $B_{R/2}$ , provided (5.8).

If (5.2) holds then we proceed similarly and we get the conclusion of Case 1.

**Case 2.**

If for the  $\theta$  found in Case 1 neither (5.2) nor (5.3) hold for any  $l \in \{1, 2\}$  then there exist  $\frac{1}{2} < \lambda_1 < 1$  and  $0 < C_0 < 1$  such that

$$\left| B_{\lambda_1 R} \cap \left\{ X_l \geq \frac{1}{2} \mu(R) \right\} \right| \leq C_0 |B_{\lambda_1 R}| \quad (5.9)$$

and

$$\left| B_{\lambda_1 R} \cap \left\{ X_l \leq -\frac{1}{2} \mu(R) \right\} \right| \leq C_0 |B_{\lambda_1 R}| \quad (5.10)$$

are satisfied for every  $l \in \{1, 2\}$ . Considering levels  $k \in [\frac{\mu(R)}{2}, \mu(R)]$ , on  $\{X_l u > k\}$  we have :

$$k^{p-2} \leq w^{\frac{p-2}{2}} \leq C_p k^{p-2}.$$

Therefore in (4.1) we can get rid of the weight:

$$\int_{B_{r'}} |\nabla_H v_l|^2 \, dx \leq \frac{C}{(r-r')^2} \int_{B_r} v_l^2 \, dx + \frac{2^{p-2} \chi}{\mu(R)^{p-2}} |A_{k,r}^+(X_l u)|^{1-\frac{2}{q}}.$$

and proceeding as in Remark 5.1, if

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{(1-\frac{Q}{q})\frac{2}{p}} \quad (5.11)$$

we get that  $X_l u \in DG^+(B_{R_0}, C_p, \chi', q)$  for levels  $k \in [\frac{\mu(R)}{2}, \mu(R)]$ , radii  $r' < r < R$  with

$$\chi' = C_p q^{\frac{12}{p}} \lambda(R_0)^2 \left( \frac{R}{R_0} \right)^{2(1-\frac{Q}{q})\frac{2}{p}} R^{2(\frac{Q}{q}-1)}.$$

Apply Lemma 3.4 with  $\lambda_1$  and  $C_0$  as in (5.9),  $k = \frac{\mu(R)}{2}$  to conclude that given  $\theta_0 \in (0, 1)$  there exists a natural number  $s = s(p, \lambda_1, C_0, \theta_0)$  such that either

$$\mu(R) \leq 2^{s+1} (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}} \quad (5.12)$$

or

$$|A_{k_s, \lambda_1 R}^+| \leq \theta |B_{\lambda_1 R}| \quad (5.13)$$

where  $k_s = \mu(R)(1 - 2^{-s-1})$ .

Now in the case (5.13) we want to use Lemma 3.3 for radii  $r' < r \in [R/2, \lambda_1 R]$ ,  $k = k_s = (1 - 2^{-s-1})\mu(R)$ ,  $\lambda_0 = 1/2$ . This can be applied if

$$k_s < \sup_{B_{\lambda_1 R}} (X_l u). \quad (5.14)$$

Then we would conclude that either

$$\begin{aligned} X_l u &\leq \frac{1}{2} k_s + \frac{1}{2} \mu(\lambda_1 R) \leq \left(1 - \frac{1}{2^{s+1}}\right) \frac{1}{2} \mu(R) + \frac{1}{2} \mu(R) \\ &= \mu(R) \left(1 - \frac{1}{2^{s+2}}\right) \quad \text{a.e. in } B_{\frac{R}{2}} \end{aligned} \quad (5.15)$$

or

$$\sup_{\lambda_1 R} (X_l u) \leq \left(1 - \frac{1}{2^{s+1}}\right) \mu(R) + (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}}. \quad (5.16)$$

If (5.14) is not true then we get

$$\sup_{B_{R/2}}(X_l u) \leq \sup_{\lambda_1 R}(X_l u) \leq k_s = \left(1 - \frac{1}{2^{s+1}}\right)\mu(R). \quad (5.17)$$

Repeating the same steps for  $-X_l u$  using assumption (5.10) and the estimate (4.2), we will find the same alternatives except instead of (5.15)-(5.17) we will have

$$X_l u \geq -\mu(R) \left(1 - \frac{1}{2^{s+2}}\right) - (\chi')^{\frac{1}{2}} R^{1-\frac{Q}{q}} \quad \text{a.e. in } B_{\frac{R}{2}}. \quad (5.18)$$

In conclusion, combining all the cases we get

$$\mu(R/2) \leq \left(1 - \frac{1}{2^{s+2}}\right)\mu(R) + c_p q^{\frac{6}{p}} 2^{s+1} \lambda(R_0) \left(\frac{R}{R_0}\right)^{\left(1-\frac{Q}{q}\right)\frac{2}{p}}.$$

■

Now we need the following technical Lemma, adapted from Lemma 7.3 in [12]:

**Lemma 5.3.** Let  $0 < A, \lambda, \alpha < 1$  with  $A \neq \lambda^\alpha$  and  $B, R_0 \geq 0$ . Let  $\varphi : [0, +\infty[ \rightarrow [0, +\infty[$  be an increasing function such that

$$\varphi(\lambda R) \leq A\varphi(R) + BR^\alpha \quad \text{for all } R \leq R_0. \quad (5.19)$$

Then for every  $R \leq R_0$  we have

$$\varphi(r) \leq \frac{1}{A} \left(\frac{r}{R}\right)^{\min\{\log_\lambda A, \alpha\}} \left[ \varphi(R) + \frac{BR^\alpha}{|A - \lambda^\alpha|} \right] \quad \text{for all } r \leq R. \quad (5.20)$$

We finally prove Theorem 1.1.

*Proof of Theorem 1.1.* We prove the result for  $u \in HW^{1,p}(\Omega)$  weak solution of the non degenerate equation (1.2). Then we can obtain the estimate for solutions to the degenerate equation (1.1) by an approximation argument as in [24] Theorem 5.3.

The alternatives in Proposition 5.2 can be combined in either

$$\mu(R/2) \leq A\mu(R) + BR^\alpha \quad (5.21)$$

or

$$|X_l u| \geq \frac{1}{32} \mu(R) \quad \text{a.e. in } B_{R/2}. \quad (5.22)$$

In this last case we have

$$w^{\frac{p-2}{2}} \geq \left(\frac{1}{32}\right)^{p-2} \mu(R)^{p-2} \quad \text{a.e. in } B_{\frac{R}{2}}.$$

Since also

$$w^{\frac{p-2}{2}} \leq \left(\delta^2 + \mu(R)^2\right)^{\frac{p-2}{2}} \leq C_p \mu(R)^{p-2} \quad \text{in } B_R,$$

from the estimate (4.1) we get

$$\int_{B_{r'}} |\nabla_{\mathbb{H}} v_l|^2 \, dx \leq \frac{C}{(r - r')^2} \int_{B_r} v_l^2 \, dx + \frac{\chi}{\mu(R)^{p-2}} |A_{k,r}^+(X_l u)|^{1-\frac{2}{q}}$$

for every  $r' < r \leq R/2$  and for every level  $k > -\mu(R_0)$ . Now as before, if

$$\mu(R) \geq \chi^{\frac{1}{p}} R^{\left(1-\frac{Q}{q}\right)\frac{2}{p}} \quad (5.23)$$

we get  $v_l \in DG^+(B_{R_0}, C_p, \chi', q)$  for every  $r' < r \leq R/2$  and for every level  $k > -\mu(R_0)$ . The same is true for  $-v_l$ , with levels  $k < \mu(R_0)$ , so proceeding as in the proof of Remark 5.1, we are in the position to apply the Oscillation Lemma 3.5 to conclude there exists  $A = A(p) \in (0, 1)$  such that

$$\text{osc}_{B_{R/4}}(X_l u) \leq A \text{osc}_{B_{R/2}}(X_l u) + BR^\alpha \leq A \text{osc}_{B_R}(X_l u) + BR^\alpha \quad \text{for all } R \leq \frac{R_0}{2}, \quad (5.24)$$

where  $B$  and  $\alpha$  are as in Proposition (5.2).

Now apply Lemma 5.3 to (5.21) and (5.24) with  $\lambda = 1/4$ ,  $A$  and  $B$  as given in (5.2). Noting that  $\text{osc}_{B_r}(X_I u) \leq 2\mu(r)$  we can combine all the estimates and hence the Theorem is proved with  $\beta = \min\{-\log_4(A), \alpha\}$ .  $\blacksquare$

**Remark 5.4.** From the explicit expression of  $\beta$  and  $B$  we see that the estimate blows up when  $q$  goes to infinity, hence the Hölder exponent found with this proof satisfies the constraint  $0 < \beta < \frac{2}{p}$ .

## 6. APPENDIX: A PROOF OF LEMMA 2.2

We use the following estimates of Zhong [26] ( see also [24], Lemmas 5.3 and 5.4).

**Lemma 6.1.** Let  $q \geq 4$  and  $\xi \in C_0^\infty(\Omega)$ . Then

$$\int_{\Omega} \xi^q w^{\frac{p-2}{2}} |Tu|^{q-2} |\nabla_{\mathbb{H}}^2 u|^2 dx \leq C_p^{\frac{q-2}{2}} (q-1)^{q-2} \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^{q-2} \int_{\Omega} \xi^2 w^{\frac{p+q-4}{2}} |\nabla_{\mathbb{H}}^2 u|^2 dx. \quad (6.1)$$

**Lemma 6.2.** Let  $q \geq 4$  and  $\xi \in C_0^\infty(\Omega)$ . Then

$$\int_{\Omega} \xi^2 w^{\frac{p+q-4}{2}} |\nabla_{\mathbb{H}}^2 u|^2 dx \leq C_p \left( \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \xi \|T\xi\|_{L^\infty} \right) (q-1)^{10} \int_{\text{supp}(\xi)} w^{\frac{p+q-2}{2}} dx.$$

Lemma 6.1 follows by using  $\phi = \xi^q |Tu|^{q-2} X_I u$  as test functions in equations (2.6) and (2.7), while Lemma 6.2 follows by using  $\phi = \xi^2 w^{\frac{q-2}{2}} X_I u$  and the estimate in Lemma 6.1.

*Proof of Lemma 2.2.* Using  $|Tu| \leq 2|\nabla_{\mathbb{H}}^2 u|$  and Lemmas 6.1 and 6.2 we have for  $q \geq 4$

$$\begin{aligned} \int_{\Omega} \xi^q w^{\frac{p-2}{2}} |Tu|^q dx &\leq 2 \int_{\Omega} \xi^q w^{\frac{p-2}{2}} |Tu|^{q-2} |\nabla_{\mathbb{H}}^2 u|^2 dx \\ &\leq C_p^{\frac{q-2}{2}} (q-1)^{q-2} \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^{q-2} \int_{\Omega} \xi^2 w^{\frac{p+q-4}{2}} |\nabla_{\mathbb{H}}^2 u|^2 dx \\ &\leq C^{\frac{q-2}{q}} (q)^{q+8} \left( \|\nabla_{\mathbb{H}} \xi\|_{L^\infty}^2 + \xi \|T\xi\|_{L^\infty} \right)^{\frac{q}{2}} \int_{\text{supp}(\xi)} w^{\frac{p+q-2}{2}} dx \end{aligned} \quad (6.2)$$

$\blacksquare$

**Note 6.3.** While working on this paper, the March 2016 preprint in the arXiv [4] was brought to my attention. This manuscript contains a general statement that includes the regularity results proved above. The proof in [4] is based on the proof put forward by Zhong in the 2009 preprint [26], that I have credited throughout. The proof in this manuscript is different than Zhong's proof, and therefore different than the proof in [4].

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